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On the penny-shaped crack in inhomogeneous elastic materials under normal extension

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Abstract

The solution of the problem of a penny-shaped crack in an inhomogeneous material with elastic coefficients which are varying continuously along the direction perpendicular to the crack is examined in this paper. We studied the problem for an inhomogeneous material which satisfies the conditions of either torsional deformation and normal extension. A series form solution to the problem is proposed and analytical expressions for the first two terms of the series are obtained by using a Hankel transform technique. In the solution a homogeneous body is chosen as the reference so that inhomogeneous quantities are treated as being perturbed from the zero's reference solutions. Closed form expressions for the relevant stress intensity factors and the crack energy are derived and specific cases of the problem are also considered.

1. Introduction

The problems determining the state of stress in the vicinity of penny-shaped cracks in a non-homogeneous solid have been discussed by (Kassir and Sih, 1975). Solutions of some mixed boundary problems in non-homogeneous materials have been solved by some authors (Kassir, 1972; Dhaliwal and Singh, 1978; Clements et al., 1978; Ergüven, 1986). Ang and Clements (1987) considered the problem for an inhomogeneous material which satisfies the conditions of either an antiplane and plane strain by using the series form solution (Kamke, 1944). Penny-shaped crack problems in an inhomogeneous elastic material under torsion have been solved by Ang (1987) and the present authors (Ergüven and Gross (1993). Gao (1991) showed that the perturbation formulae can be derived from the potential energy bounds for nonhomogeneous materials, and applied the

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perturbation algorithm to calculate the stress intensity factors for several crack problems involving spatially varying material moduli.

Recently, Craster and Atkinson (1994) studied some mixed boundary value problem for inhomogeneous elastic materials where the shear modulus varies with respect to one space variable as $(a+b|z|)^n$, where n is not necessarily an integer. They use Fourier transforms and the Wiener–Hopf technique to solve antiplane and plane strain problems.

All the above authors have considered crack problems in which non-homogeneity is unidirectional and crack is on the plane of the material symmetry. As indicated in all the studies, as long as the crack tip is embedded in a homogeneous medium, the stress state around the crack tip would have the standard square-root singularity and the conventional methods of fracture mechanics would be applicable. But for a class of non-homogeneity, the power of stress singularity is real and not equal to 1/2. Although this result is physically acceptable, it is not readily suitable for conventional fracture mechanics applications. Also in this case the coefficient of stress intensity can no longer be interpreted as a stress intensity factor in the usual way.

In this paper, we studied the problem of determining the state of stress in an infinite non-homogeneous elastic medium containing a penny-shaped crack under torsional (Mode III) and normal stresses (Mode I). The shear modulus of the material is assumed to vary slightly in the normal direction to the crack, while Poisson's ratio remains constant throughout the material. The spatial variation of the shear modulus μ is assumed to be of the form $\mu = \mu_0+\varepsilon f(z)$ and that the variation of the shear modulus is slow along the direction normal to the crack surface. The shear modulus changes according to the parameters such that $\varepsilon \ll 1$ and f is a given function differentiable of z. A series form solution to the problem are proposed and the first two terms of the series are obtained by using the Hankel transform technique. For simple variation of shear modulus, the close form analytical solutions are given both problems and the dependence to the material constants on the singularity are obtained.

2. Statement and formulation of the problem

We will use a cylindrical coordinate system (r, ϑ, z) with the z axis perpendicular to the crack plane. The displacement equation of equilibrium for a non-homogeneous, isotropic and elastic solid is given by

$$\mu\nabla\cdot\nabla\mathbf{u}+(\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u}+(\nabla\mathbf{u}+(\nabla\mathbf{u}))^T\nabla\mu+(\mathrm{div}\,\mathbf{u})\nabla\lambda = 0 \tag{1}$$

where λ and μ are the Lamé parameters and $\mathbf{u}$ is the displacement vector. The clear expression of this equation in cylindrical coordinates is given in Ergüven (1986). The stresses are related to the strains by

$$\sigma_{ij} = 2\mu e_{ij}+\lambda\delta_{ij}e_{kk} \tag{2}$$

where δ_{ij} is the Kronecker delta and the strains e_{ij} are defined as

$$2e_{ij} = u_{i,j}+u_{j,i} \tag{3}$$

where u_i is Cartesian components of displacements. If we choose

$$u_r = u_r(r,z), \quad u_\vartheta = u_\vartheta(r,z), \quad u_z = u_z(r,z), \quad \mu = \mu(z), \quad \lambda = \text{constant} \tag{4}$$

eqn (1) can be written as

$$\mu\left[Su_r + \frac{1}{(1-2v)}D_r e\right] + D\mu D(u_r + u_z) = 0 \tag{5}$$

$$\mu S u_\vartheta + D\mu D u_\vartheta = 0 \tag{6}$$

$$\mu\left[\nabla^2 u_z + \frac{1}{(1-2v)}De\right] + 2D\mu\left[Du_z + \frac{v}{(1-2v)}e\right] = 0 \tag{7}$$

in which ∇^2 is the Laplacian in the cylindrical coordinate given by

$$\nabla^2 = \partial^2/\partial r^2 + (1/r)\,\partial/\partial r + \partial^2/\partial z^2 \tag{8}$$

and

$$Su_r = \nabla^2 u_r - \frac{u_r}{r^2}, \quad D_r = \partial/\partial r, \quad D = \partial/\partial z \tag{9}$$

In eqns (5) and (7), v is the Poisson's ratio of the material and e is the dilatation which can be expressed in terms of the displacement components as

$$e = \partial u_r/\partial r + u_r/r + \partial u_z/\partial z \tag{10}$$

2.1. *Torsional stresses*

The axisymmetric torsion problem has been considered by using the same method in Ergüven and Gross (1993) and here it is given short formulation. The Navier equilibrium equation for torsion problem is given in eqn (6). The spatial variation of the shear modulus μ is assumed to be of the form

$$\mu = \mu_0 + \varepsilon f(z) \tag{11}$$

where μ_0 is a constant, ε is a constant parameter such that $\varepsilon \ll 1$ and f is a given function is continuous and differentiable. Substituting eqn (11) into eqn (6), we obtain

$$\mu\nabla^2 u_\vartheta + \varepsilon D f D u_\vartheta = 0 \tag{12}$$

Proposing a solution to eqn (12) in the form

$$u_\vartheta = \sum_0^\infty \varepsilon^n u_n(r,z) \tag{13}$$

and substituting eqn (13) into eqn (12) and expanding the resulting expression in a power series, then equating the coefficient same power of ε to zero, we obtain

$$\nabla^2 u_0 = 0 \tag{14}$$

and recurriance relation for the other solutions

$$\mu_0 \nabla^2 u_n = -[DfDu_{n-1} + f(z)\nabla^2 u_{n-1}], \quad n > 0 \tag{15}$$

The non-zero components of the stress tensor are given by

$$\sigma_{\vartheta z} = \mu D u_\vartheta, \quad \sigma_{r\vartheta} = \mu r D_r (u_r/r) \tag{16}$$

From eqns (13) and (16) we obtain

$$\sigma_{\vartheta z} = \sigma^0_{\vartheta z} + \varepsilon \sigma^1_{\vartheta z} + O(\varepsilon^2) \tag{17}$$

where

$$\sigma^0_{\vartheta z} = \mu_0 D u_0 \tag{18}$$

$$\sigma^1_{\vartheta z} = \mu_0 D u_1 + f(z) D u_0 \tag{19}$$

2.2. *Axisymmetric normal stresses*

Let the penny-shaped crack be subjected to normal tractions that vary in the radial direction only and give rise to a stress system independent of ϑ. For the problem described by eqn (4), the appropriate Navier equations in terms of the displacements u_r and u_z are given by eqns (5) and (7). We choose the shear modulus as given in eqn (11). If we propose a solution to eqns (5) and (7) in the form given in eqn (13)

$$u_r = \sum_0^\infty \varepsilon^n u_r^{(n)}, \quad u_z = \sum_0^\infty \varepsilon^n u_z^{(n)} \tag{20}$$

then from eqn (10) we may write

$$e = \sum_0^\infty \varepsilon^n e^{(n)}(r, z) \tag{21}$$

Substituting eqns (20) and (21) into eqns (5) and (7), if we are interested in only the first two terms of series solutions of eqn (20), we obtain that it is only necessary to solve in the following equations,

$$Su_r^0 + \frac{1}{(1-2v)} D_r e^0 = 0 \tag{22}$$

$$\nabla^2 u_z^0 + \frac{1}{(1-2v)} De^0 = 0 \tag{23}$$

and

$$Su_r^1 + \frac{1}{(1-2v)} D_r e^1 = -\mu_0^{-1} DfD(u_r^0 + u_z^0) \tag{24}$$

$$\nabla^2 u_r^1 + \frac{1}{(1-2v)} De^1 = -2\mu_0^{-1} Df \left[Du_z^0 + \frac{v}{(1-2v)} e^0 \right] \tag{25}$$

Using eqn (2), stresses can be written in the following form

$$\sigma_{ij} = \sigma^0_{ij} + \varepsilon\sigma^1_{ij} + O(\varepsilon^2) \tag{26}$$

where

$$\sigma^0_{ij} = 2\mu_0 e^0_{ij} + \lambda\delta_{ij} e^0_{kk} \tag{27}$$

$$\sigma^1_{ij} = 2\mu_0 e^1_{ij} + f e^0_{ij} + \lambda\delta_{ij} e^1_{kk} \tag{28}$$

2.3. *Stress intensity factors for penny-shaped crack problems*

In the analysis of crack problems, it is essential to have a knowledge of the asymptotic behaviour of the stresses around the crack border. The coefficients K_{I}, K_{II}, K_{III} which are commonly known as the stress intensity factor are dependent on the crack geometry, loading conditions and non-homogeneity parameters. We are interested here in calculating the stress intensity factors K_{I}, K_{II} and K_{III} are defined by

$$K_{\mathrm{I}} = \lim_{r\to a^+} [2(r-a)]^{1/2}\sigma_z(r,z) \tag{29a}$$

$$K_{\mathrm{II}} = \lim_{r\to a^+} [2(r-a)]^{1/2}\sigma_{rz}(r,z) \tag{29b}$$

$$K_{\mathrm{III}} = \lim_{r\to a^+} [2(r-a)]^{1/2}\sigma_{\vartheta z}(r,z) \tag{30}$$

From eqn (24), K_i, $i = \mathrm{I, II, III}$ may be written as

$$K_i = K^0_i + \varepsilon K^1_i + O(\varepsilon^2) \tag{31}$$

where K^0_i is defined by using σ^0_{ij} and K^1_i is defined by using σ^1_{ij} in eqns (29) and (30).

The Mode III penny-shaped crack problem in the infinite elastic solid has been solved by the present authors using the same method (Ergüven and Gross, 1993). Consider the case of $f(z) = k|z|$, k is a positive constant and the shear stress on the crack surface $\tau(r) = \tau_0 r/a$, the stress intensity factors are

$$K^0_{\mathrm{III}} = 4\tau_0\sqrt{a}/3\pi \tag{50}$$

$$K^1_{\mathrm{III}} = -\frac{\tau_0\sqrt{a}k}{6\mu_0\pi\sqrt{\pi}\Gamma(3/2)} \tag{53}$$

Neglecting $O(\varepsilon^2)$ terms, the stress intensity factor was obtained as follows:

$$K_{\mathrm{III}} = \frac{4}{3\pi}\tau_0\sqrt{a}\left(1 - \frac{\varepsilon k a}{2\pi\mu_0\Gamma(3/2)}\right) \tag{54}$$

This result clearly shows that for this particular type of inhomogeneous material with shear modulus which increases with $|z|$ the stress intensity factor for a material with constant shear modulus μ_0. Furthermore, by decreasing the value of μ_0, the difference between these stress intensity factors becomes more pronounced (Ergüven and Gross, 1993).

3. Normal stress in nonhomogeneous elastic media and Plevako's formulation

In this section, we consider the problem of determining the stress distribution in the vicinity of a penny-shaped crack in an infinite isotropic non-homogeneous material. Poisson's ratio v of the material is assumed to be constant while the shear modulus varies as in eqn (11). Following Plevako (1972) the governing equation can be written as

$$\nabla^2\left(\frac{1}{\mu}\nabla^2 L\right)-\frac{1}{1-v}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)L\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{\mu}\right)=0 \tag{32}$$

where L is a function which reduces to the biharmonic equation in the homogeneous case. Stresses and displacement can be expressed by using L function as follows:

$$\sigma_z=\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)^2 L,\quad \tau=-\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z} \tag{33}$$

$$u_z=-\frac{1}{\mu}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z}+\frac{\partial}{\partial z}\left[\frac{1}{2\mu}\left(v\nabla^2 L-\frac{\partial^2 L}{\partial z^2}\right)\right] \tag{34}$$

We propose a solution to eqn (32) in the form

$$L=\sum_{n=0}^{\infty}\varepsilon^n L_n(r,z) \tag{35}$$

By using eqns (11), (32) and (35) we find that it is necessary to solve for only two terms of the series solution,

$$\nabla^2\nabla^2 L_0=0 \tag{36}$$

and

$$\nabla^2\nabla^2 L_1=q/\mu_0 \tag{37}$$

where

$$q=2f'\frac{\partial}{\partial z}\nabla^2 L_0-\frac{1}{1-v}f''(v\nabla^2 L_0-\mathrm{d}^2 L_0/\mathrm{d}^2 z) \tag{38}$$

Using eqns (11), (34) and (35), and assuming $|\varepsilon f/\mu_0| \ll 1$, the normal displacement u_z can be written as

$$u_z=u_z^0+\varepsilon u_z^1+O(\varepsilon^2) \tag{39}$$

where

$$u_z^0=-\frac{1}{\mu_0}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\frac{\partial L_0}{\partial z}+\frac{1}{2\mu_0}\frac{\partial}{\partial z}\left(v\nabla^2 L_0-\frac{\partial^2 L_0}{\partial z^2}\right) \tag{40}$$

$$u_z^1 = \frac{1}{\mu_0}\left\{\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L_1}{\partial z} + \frac{1}{2\mu_0}\frac{\partial}{\partial z}\left(v\nabla^2 L_1 - \frac{\partial^2 L_1}{\partial z^2}\right)\right\} - \frac{f(z)}{\mu_0}u_z^0 - \frac{f'}{2\mu_0^2}\left(v\nabla^2 L_0 - \frac{\partial^2 L_0}{\partial z^2}\right) \quad (41)$$

Stresses may also be written by making use of eqns (33) and (35) as

$$\sigma_z = \sigma_z^0 + \varepsilon\sigma_z^1 + O(\varepsilon^2) \quad (42)$$

$$\tau_{rz} = \tau_{rz}^0 + \varepsilon\tau_{rz}^1 + O(\varepsilon^2) \quad (43)$$

where

$$\sigma_z^i = \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)L_i, \quad (\text{for } i = 0, 1) \quad (44)$$

$$\tau_{rz}^i = -\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L_i}{\partial z}, \quad (\text{for } i = 0, 1) \quad (45)$$

Consider an infinite elastic material whose shear modulus μ is given by eqn (11) with f being an even function of z. The solid contains a penny-shaped crack in the region $0 < r < a$, $z = 0$, pressurized by symmetric normal stress $\sigma_z = \sigma_0(r)$ and $\tau_{rz} = 0$ act on the crack. Due to the symmetry of the problem about $z = 0$ plane, the problem described above is equivalent to the problem of solving eqn (32) subject to the boundary conditions

$$\tau_{rz} = 0 \quad \text{for all values of } r, \quad z = 0 \quad (46)$$

$$\sigma_z = \sigma_0(r) \quad \text{for } 0 \leqslant r < a, \quad z = 0 \quad (47)$$

$$u_z = 0 \quad \text{for } r > a, \quad z = 0 \quad (48)$$

If we use the first two terms of eqn (35), this boundary value problem may be replaced by Problems 3.1 and 3.2 below.

Problem 3.1

Solve eqn (36) subject to

$$\tau_{rz}^0 = 0 \quad \text{for all values of } r, \quad z = 0 \quad (49)$$

$$\sigma_z^0 = \sigma_0(r) \quad \text{for } 0 \leqslant r < a \quad \text{and} \quad u_z^0 = 0 \quad \text{for } r > a; \quad z = 0 \quad (50)$$

Problem 3.2

Solve eqn (37) subject to

$$\tau_{rz}^1 = 0 \quad \text{for all values of } r, \quad z = 0 \quad (51)$$

$$\sigma_z^1 = 0 \quad \text{for}\, 0 \leqslant r < a \quad \text{and} \quad u_z^1 = 0 \quad \text{for}\, r > a\,; \quad z = 0 \tag{52}$$

Solution of Problem 3.1

Now we can write from the solution satisfied regularity condition of eqn (36), satisfying the boundary condition (46) and regularity condition at infinity as

$$\mathscr{L}_0(\xi, z) = A(\xi)(1+\xi z)\exp(-\xi z) \tag{53}$$

where $\mathscr{L}_0$ is the Hankel transform and ξ is the Hankel transform parameter. Making use of eqns (40), (44) and (53)

$$2\mu_0 u_z^0(\xi, z) = -A(\xi)[2(1-v)+\xi z]\xi^3 \exp(-\xi z) \tag{54}$$

$$\sigma_z^0(\xi, z) = A(\xi)\xi^4(1+\xi z)\exp(-\xi z) \tag{55}$$

and inserting the boundary conditions (50) the problem reduces to that of solving the dual integral equations

$$\int_0^\infty \xi \bar{A}(\xi) J_0(\xi r)\,\mathrm{d}\xi = p(r), \quad 0 \leqslant r < a \tag{56}$$

$$\int_0^\infty \bar{A}(\xi) J_0(\xi r)\,\mathrm{d}\xi = 0, \quad r > a \tag{57}$$

where $\bar{A}(\xi) = \xi^3 A(\xi)$.

The solution of above dual integral equation may be written as (Sneddon, 1966)

$$\bar{A}(\xi) = \frac{2}{\pi}\int_0^a \sin(\xi t)\,\mathrm{d}t \int_0^t r p(r)(t^2-r^2)^{-1/2}\,\mathrm{d}r \tag{58}$$

and the stress intensity factor is

$$K_1^0 = \frac{2}{\pi} a^{-1/2} \int_0^a r p(r)(a^2-r^2)^{-1/2}\,\mathrm{d}r \tag{59}$$

Solution of Problem 3.2

The function $L_1(r, z)$ defined by

$$L_1(r, z) = \int_0^\infty \xi G(\xi, z) J_0(\xi r)\, e^{-\xi z}\,\mathrm{d}\xi \tag{60}$$

is a solution of eqn (37) if the function $G(\xi, z)$ satisfies

$$G^{\mathrm{IV}} - 4\xi G''' + 4\xi^2 G'' = \frac{A(\xi)\xi^2}{\mu_0}\left[4f'\xi + \frac{f''}{1-v}(2v-1+\xi z)\right] \tag{61}$$

The general solution of eqn (61) is

$$G(\xi,z) = G_p(\xi,z)+C_1+C_2z+C_3\exp(2\xi z)+C_4\exp(-2\xi z) \tag{62}$$

where C_Y, $i = 1$–4, are arbitrary functions of ξ and $G_P(\xi,z)$ is given by (Kamke, 1944)

$$G_p(\xi,z) = -e^{-2\xi z}\int^z tW(\xi,t)\,e^{-2\xi t}\,\mathrm{d}t+z\,e^{2\xi z}\int^z W(\xi,t)\,e^{-2\xi t}\,\mathrm{d}t \tag{63}$$

where $W(\xi,z)$ is defined by

$$W(\xi,z) = \frac{A(\xi)\xi^2}{\mu_0(1-v)}\left[2\xi(1-2v)\int^z f(t)\,\mathrm{d}t+(2v-1+\xi z)f(z)\right] \tag{64}$$

Since we require the displacements and stresses to vanish at infinity, it is necessary to set the function C_3 and C_4 to zero. The use of condition (51) yields

$$\mathrm{d}G_p/\mathrm{d}z|_{z=0}-\xi G_p(\xi,0) = C_1\xi-C_2 \tag{65}$$

If we assume that the stress σ^1_{zz} is such that

$$\sigma^1_{zz}(r,0) = p(r) \tag{66}$$

then from eqns (44)–(59) and through the use of Hankel inversion theorem we obtain

$$C_1(\xi) = \gamma(\xi)/\xi^4-G_p(\xi,0) \tag{67}$$

where $\gamma(\xi)$ is defined by

$$\gamma(\xi) = \int_0^\infty rp(r)J_0(\xi r)\,\mathrm{d}r \tag{68}$$

From eqns (65) and (67), C_2 is given by

$$C_2(\xi) = \gamma(\xi)/\xi^3-(\partial G_p/\partial z)|_{z=0} \tag{69}$$

Making use of eqns (41)–(60) and (63) we can obtain

$$u^1_z(r,0) = \frac{(v-1)}{2\mu_0}\int_0^\infty \xi\left[\frac{2\gamma(\xi)}{\xi}+X(\xi)+Z(\xi)A(\xi)\right]J_0(\xi,r)\,\mathrm{d}\xi \tag{70}$$

$$\sigma^1_{zz}(r,0) = \int_0^\infty \xi\gamma(\xi)J_0(\xi r)\,\mathrm{d}\xi \tag{71}$$

where

$$X(\xi) = G'''_p-3\xi G''_p \tag{72}$$

$$Z(\xi) = -2f(0)\xi^3/\mu_0+\frac{(1-2v)f'(0)}{(1-v)\mu_0}\xi^2 \tag{73}$$

The task now is to determine $\gamma(\xi)$ by using the remaining boundary condition, namely condition (52). Making use of eqns (70) and (71), the condition (52) yields

$$\int_0^\infty \xi\gamma(\xi)J_0(\xi r)\,\mathrm{d}\xi = p(r)\,; \quad r < a \tag{74}$$

$$\int_0^\infty [2\gamma(\xi)+\xi X(\xi)+\xi Z(\xi)A(\xi)]J_0(\xi r)\,\mathrm{d}\xi = 0\,; \quad r > a \tag{75}$$

By defining the new function as

$$C(\xi) = 2\gamma(\xi)+\xi X(\xi)+\xi Z(\xi)A(\xi) \tag{76}$$

The solution of above dual integral equation for new unknown $C(\xi)$ is given by (Sneddon, 1966)

$$C(\xi) = \frac{1}{\pi}\int_0^a \sin(\xi t)\,\mathrm{d}t \int_0^t rq(r)(t^2-r^2)^{-1/2}\,\mathrm{d}r \tag{77}$$

where, with $p(r) = 0$,

$$q(r) = \int_0^\infty \xi[\xi X(\xi)+\xi Z(\xi)A(\xi)]J_0(\xi r)\,\mathrm{d}\xi \tag{78}$$

From eqn (77) with integration by parts and eqn (71) we obtain

$$K_\mathrm{I}^1 = -\frac{2}{\pi}\frac{F(a)}{\sqrt{\xi}} - \frac{1}{2}\lim_{r\to a}(r-a)\int_0^\infty \xi^2[X(\xi)+Z(\xi)A(\xi)]J_0(\xi r)\,\mathrm{d}\xi \tag{79}$$

where

$$F(a) = \int_0^a rq(r)(a^2-r^2)^{-1/2}\,\mathrm{d}r \tag{80}$$

Example: *Uniform pressure*

If a uniform pressure $\sigma_0(r) = -\sigma_0$ (constant) acts on the crack then eqn (58)

$$\bar{A}(\xi) = a^{3/2}\sigma_0(2\pi\xi)^{-1/2}J_{3/2}(a\xi) \tag{81}$$

and from eqn (59) the stress intensity factor K_I^0 is

$$K_\mathrm{I}^0 = \frac{2}{\pi}\sigma_0\sqrt{a} \tag{82}$$

We now consider the case where shear modulus μ is given by

$$\mu(z) = \mu_0+\varepsilon kz \tag{83}$$

where k is a positive constant. From eqns (63), (64) and (83) and differentiating, we obtain

$$X(\xi) = \frac{3k}{\mu_0}\xi^2 A(\xi) \tag{84}$$

Taking $f(0) = 0$ and $f'(0) = k$ into account, from eqn (73) we obtain

$$Z(\xi) = \frac{(1-2\nu)k}{(1-\nu)\mu_0}\xi^2 \tag{85}$$

Substituting eqns (81), (84) and (85) into eqns (81) and (80) we obtain the stress intensity factor K_1 as follows:

$$K_{\mathrm{I}}^1 = -\frac{2ka^{3/2}}{\pi^2\mu_0}\sigma_0\left(3+\frac{2\nu+1}{1-\nu}\right) \tag{86}$$

Making use of eqns (29a), (82) and (86), in the case of $O(\varepsilon^2)$, the stress intensity factor is given by

$$K_{\mathrm{I}} = \frac{2}{\pi}\sigma_0\sqrt{a}\left[1-\frac{3\varepsilon ka}{\pi\mu_0}-\frac{\varepsilon ka}{\pi\mu_0}\frac{(2\nu-1)}{(1-\nu)}\right] \tag{87}$$

We can calculate the crack surface displacement by using eqn (62). From the solution of Problem 3.1 for the constant pressure. We can obtain from eqn (54) and (58) the surface displacement as

$$u_z^0(r,0) = [2(1-\nu)\sigma_0/(\pi\mu_0)](a^2-r^2)^{1/2} \tag{88}$$

The second term in eqn (39) may be obtained from the solution Problem 3.2. Using the eqn (70) and substituting eqns (58), (72), (73) and (77) we obtain

$$u_z^1(r,0) = -u_z^0(r,0)\frac{ka}{\pi\mu_0}\frac{(2-\nu)}{(1-\nu)} \tag{89}$$

As a result the crack surface displacement may be written as

$$u_z(r,0) = u_z^0(r,0)\left[1-\frac{\varepsilon ka}{\pi\mu_0}\frac{(2-\nu)}{(1-\nu)}\right] \tag{90}$$

To calculate the elastic energy W expended in forming the crack, use is made of the formula

$$W = 2\int_0^a p(r)u_z(r,0)\,\mathrm{d}r \tag{91}$$

If we substitute the form (187) for $u_z(r,0)$, in the case in which $p(r) = \sigma_0$ this gives

$$W = [4(1-\nu)\sigma_0^2 a^3/3\mu_0]\left[1-\frac{\varepsilon ka}{\pi\mu_0}\frac{(2-\nu)}{(1-\nu)}\right] \tag{92}$$

The energy release rate may also be written as

$$G = [4(1-\nu)\sigma_0^2 a/\mu_0\pi]\left[1-\frac{\varepsilon ka}{\pi\mu_0}\frac{(2-\nu)}{(1-\nu)}\right] \tag{93}$$

4. Results and discussion

The stress intensity factor (31) for the normal extension of penny-shaped crack indicates behaviour which is qualitatively consistent with the corresponding result (Kassir, 1972; Ergüven, 1986; Ergüven and Gross, 1993; Ang, 1987). As is well known the stress intensity factors for the plane case and the penny-shaped crack differ with the constant $2/\pi$ in the case of homogeneous medium and of the stress acting on the crack is constant. If we compare our solution (87) and the results obtained by (Ang and Clements, 1987) for the plane case, we can see that the results differ with the constant $2/\pi$ as that is the homogeneous case.

The stress intensity factor for the penny-shaped crack in a material with variable shear modulus is less than the corresponding factor for constant shear modulus μ_0 for $\varepsilon > 0$. As is seen easily the magnitude of difference between the homogeneous and inhomogeneous stress intensity factors decrease as μ_0 increases.

In homogeneous spaces the stress intensity factor is independent of μ_0 and is given in eqn (82), whereas the crack surface displacement is inversely proportional to μ_0 as seen in eqn (88). In the homogeneous case the stress intensity factor is dependent of $\mu(0) = \mu_0$ and less than the corresponding factor for a material with shear modulus μ_0 for $\varepsilon > 0$. The magnitude of these two stress intensity factors decreases as μ_0 increases. From eqn (90) it is seen that the crack surface displacement $u_z(r, 0)$ for the inhomogeneous medium is less than that for the homogeneous medium having the modulus $\mu(0) = \mu_0$. Since the stress intensity factor is related to the magnitude of the crack surface displacement derivative, it would, therefore be, expected that the stress intensity factor for the inhomogeneous medium would be smaller than $2\sigma_0\sqrt{a/\pi}$, the value for the corresponding homogeneous medium.

It can be pointed out from eqn (87) that the stress intensity factor is still related to the inhomogeneity parameters in the case of incompressible materials. Since for compressible materials Poisson's ratio ν satisfies $0 < \nu < 1/2$, from eqn (87) the magnitude of the difference between the stress intensity factors for the homogeneous and inhomogeneous materials is bounded above by $3\sigma_0\varepsilon ka^{3/2}/\pi\mu_0$ and below by $2\sigma_0\varepsilon ka^{3/2}/\pi\mu_0$.

We have evaluated the stress intensity factor for some values of ε and ν. Table 1 shows the stress intensity factor for different ε and $\nu = 0.2, 0.3, 0.4$. Also in Fig. 1 we graphed the stress intensity factor.

With the increasing modulus of rigidity in the neighbourhood of the crack, the stress intensity factors are shown to decrease in this work for $\varepsilon > 0$ and the results seem to confirm the results of Kassir (1972), Dhaliwal and Singh (1978), Ang (1987). It is possible to obtain the case of a decrease in the modulus of rigidity in the vicinity of the crack $\varepsilon < 0$. The stress intensity factors are increased with the decrease μ the modulus of rigidity in the neighbourhood of the crack in the case of the constant nonhomogeneity parameters.

The results obtained in the present article are very similar those to obtained by Gao (1991), although methods of solutions are different.

Table 1

	K_I/K_I^0											
$ka/\pi\mu_0$		0	1	2	3	4	5	6	7	8	9	10
$\varepsilon = 0.00$		1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\varepsilon = 0.05$	$v = 0.0$	1	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
	$v = 0.5$	1	0.85	0.70	0.55	0.40	0.25	0.10				
$\varepsilon = 0.10$	$v = 0.0$	1	0.80	0.60	0.40	0.20	0.00					
	$v = 0.5$	1	0.70	0.40	0.10							

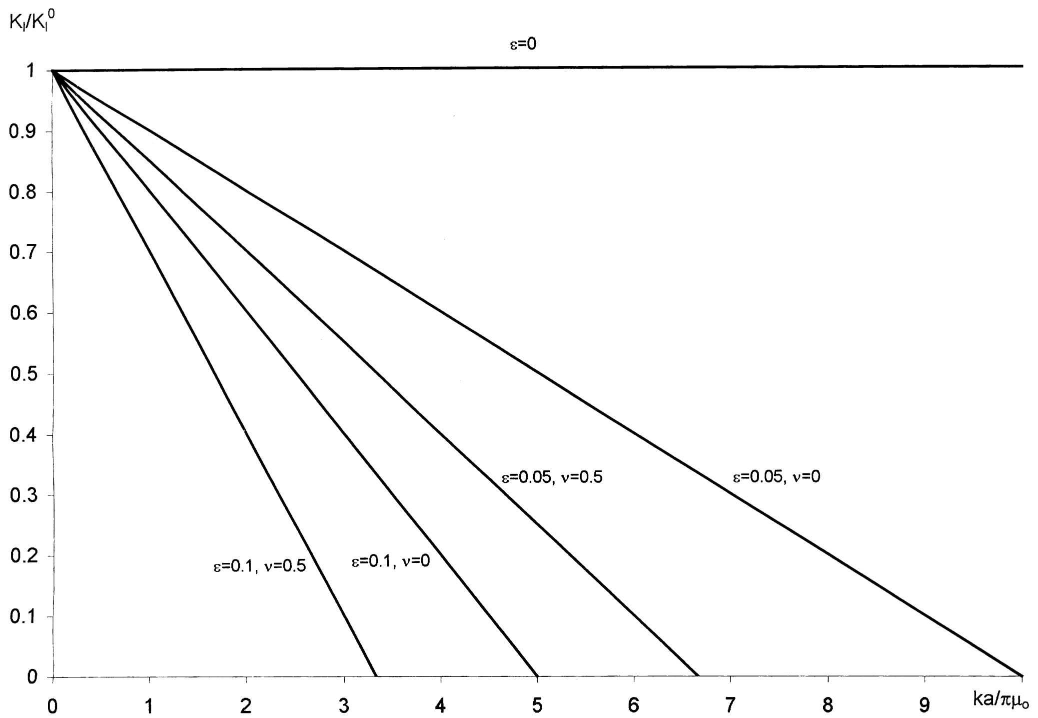

Fig. 1. The mode I stress-intensity factor as a function of $ka/\pi\mu_0$ for different values of ε and v.

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References

Ang, W.T., 1987. A penny-shaped crack in an inhomogeneous elastic material under axisymmetric torsion. Solid Mechanics Archives 12(2), 391–422.

Ang, W.T., Clements, D.L., 1987. On some crack problems for inhomogeneous elastic materials. International Journal of Solids and Structures 23(8), 1089–1104.

Clements, D.L., Atkinson, C., Rogers, C., 1987. Antiplane crack problems for an inhomogeneous elastic material. Acta Mechanica 29, 199.

Craster, R.V., Atkinson, C., 1994. Mixed boundary value problems in non homogeneous elastic materials. Quaterly Journal of Mechanics and Applied Mathematics 47, 183–206.

Dhaliwal, R.S., Singh, B.H., 1987. On the theory of elasticity of a nonhomogeneous medium. Journal of Elasticity 8, 211.

Ergüven, M.E., 1986. Penny-shaped cracks in media with spatially varying elastic modulus. Engineering Fracture Mechanics 24, 25–31.

Ergüven, M.E., Gross, D., 1993. On the penny-shaped crack in inhomogeneous elastic materials. ZAMM 74, 179–182.

Gao, H., 1991. Fracture analysis of non-homogeneous materials via a moduli-perturbation approach. International Journal of Solids and Structures 27(13), 1663–1682.

Kamke, E., 1944. Differantial Gleichungen (Losungmethoden und Losungen). Band 1, Becker@ErlerKom.-Ges.

Kassir, M.K., 1972. A note on the twisting deformation of a nonhomogeneous shaft containing a circular crack. International Journal of Fracture Mechanics (8), 325–334.

Kassir, M.K., Sih, G.C., 1975. Three Dimensional Crack Problems, ed. G. C. Sih. Noordhoff, Leyden.

Plevako, V.P., 1972. On the theory of elasticity of inhomogeneous media. PMM 35(5), 853–860.

Sneddon, I.N., 1966. Mixed Boundary Value Problems in Potential Theory. North-Holland, Amsterdam.